A Lower Bound for Chaos on the Elliptical Stadium

Eduardo Canale*   Roberto Markarian⋆

Inst. de Matemática y Estadística "Prof. Ing. Rafael Laguardia"

Fac. de Ingeniería, Universidad de la Republica

C.C. 30, Montevideo, Uruguay.

Sylvie Oliffson Kamphorst⋄   Sônia Pinto de Carvalho°

Departamento de Matemática, ICEx, UFMG

C.P. 702, 30161–970, Belo Horizonte, Brasil.

Abstract

The elliptical stadium is a plane region bounded by a curve constructed by joining two half-ellipses, with half axes $a > 1$ and $b = 1$, by two parallel segments of equal length $2h$.

V. Donnay [2] proved that if $1 < a < \sqrt{2}$ and if h is big enough than the corresponding billiard map has non-vanishing Lyapunov exponents almost everywhere; moreover $h \to \infty$ as $a \to \sqrt{2}$. In a previous paper [4] we found a bound for h assuring the K-property for these billiards, for values of a very close to 1.

In this work we study the stability of a particular family of periodic orbits obtaining a new bound for the chaotic zone for any value of $a < \sqrt{2}$.

1   Introduction

The elliptical stadium is a plane region bounded by a curve Γ, constructed by joining two half-ellipses, with major axes $a > 1$ and minor axes $b = 1$, by two straight segments of equal length $2h$ (see fig.1).
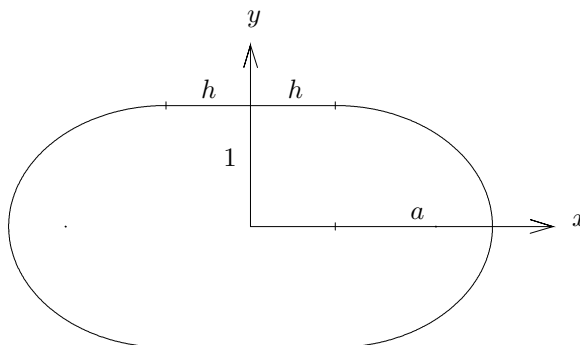

Figure 1: The elliptical stadium.

*canale@fing.edu.uy   ⋆roma@fing.edu.uy   ⋄syok@mat.ufmg.br   °sonia@mat.ufmg.br

The billiard on the elliptical stadium consists in the study of the free motion of a point particle inside the stadium, being reflected elastically at the impacts with Γ. Since the motion is free inside Γ, it is determined either by two consecutive points of reflection at Γ or by the point of reflection and the direction of motion immediately after each collision.

Let $s \in [0, L)$ be the arclength parameter for Γ and the direction of motion be given by the angle β with the normal to the boundary at the impact point. The billiard defines a map T from the annulus $\mathcal{A} = [0, L) \times (-\pi/2, \pi/2)$ into itself. Let (s_0, β_0) and $(s_1, \beta_1) \in \mathcal{A}$ be such that $T(s_0, \beta_0) = (s_1, \beta_1)$ and that Γ is C^∞ in some neighborhood of s_0 and s_1 (notice that Γ is globally C^1 but not C^2 and piecewise C^∞). Then, T is a C^∞-diffeomorphism in some neighborhoods of (s_0, β_0) and (s_1, β_1). It also preserves the measure $d\mu = \cos\beta \, d\beta \, ds$ (see, for instance, [3]).

$(\mathcal{A}, \mu, T)$ defines a discrete dynamical system, whose orbits are given by

$$\mathcal{O}(s_0, \beta_0) = \{(s_n, \beta_n) = T^n(s_0, \beta_0), n \in Z\} \subset A.$$

So the elliptical stadium billiard defines, almost everywhere, a two-parameter family of diffeomorphisms $T_{a,h}$ whose dynamics depend on the values of a and h. For instance, when $h = 0$, $T_{a,h}$ is integrable for every a since we have the elliptical billiard.

In [2], V. Donnay proved that the elliptical billiard stadium map $T_{a,h}$ has non-vanishing Lyapunov exponents almost everywhere if $1 < a < \sqrt{2}$ and h is sufficiently large. He also proved that h must go to infinity as a approaches $\sqrt{2}$. Donnay addressed a challenge: "One could try to calculate bounds on these lengths."

In [4], we proved that if $1 < a < \sqrt{4 - 2\sqrt{2}}$, then $h > 2a^2\sqrt{a^2 - 1}$ assures not only the positiveness of a Lyapunov exponent, but also ergodicity and the K-property. However, $2a^2\sqrt{a^2 - 1}$ does not seem to be an optimal lower bound for h. Numerical simulations exhibit chaotic phase spaces for values of h smaller than this bound. Moreover, $\sqrt{4 - 2\sqrt{2}} \approx 1.082$ is far from $\sqrt{2}$ and close to 1, so in this case we are very close to the Bunimovich stadium which is *chaotic* for all $h > 0$.

In this work, expanding and revising the work in [1], we study a family of periodic orbits, with any pair period $p \geq 4$, whose behavior looks generic. This means that in phase space, when they are elliptic, they are surrounded by invariant curves which constitute *elliptic islands* of positive measure which disappear as the orbits change from elliptic to hyperbolic.

We will prove the existence of a curve $H = H(a)$, which diverges as a approaches $\sqrt{2}$ such that above it all these periodic orbits are hyperbolic. Even though $H = H(a)$ may not be the optimal searched bound for chaoticity, it is at least a lower bound and seems not far from it. It also gives a very good answer to Donnay's challenge.

This paper continues in the following way: in section 2 we describe the family of periodic orbits. In section 3, we study their hyperbolicity. Section 4 consists on the definition of the bound $H(a)$. Section 5 contains some concluding remarks.

2   Pantographic Orbits

On $\mathbb{R}^2$, we fix the origin on the center of the elliptical stadium and take the x-axes containing the major half-axis of the half-ellipses (see figure 1).

Given a, h and a positive integer i, an (i, a, h)-pantographic orbit, denoted by $Pan(i, a, h)$, is a $(4 + 2i)$-periodic orbit, symmetric with respect to the coordinate axes, alternating the impacts up and down, with

exactly 4 impacts at the half-ellipses (2 at each one, joined by a vertical path) and $2i$ impacts at the straight parts (i at each one) and crossing the y-axis only twice (see figure 2).
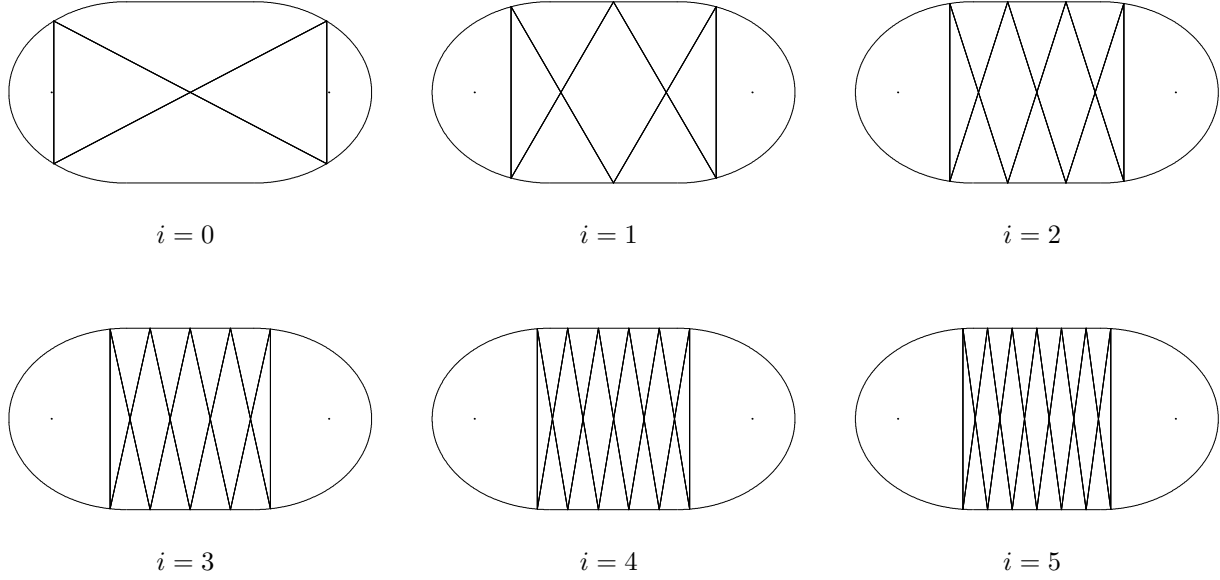

Figure 2: Pantographic orbits of the elliptical stadium.

The choice of those pantographic orbits was motivated by several remarks. The elliptical stadium billiard can be viewed as a perturbation of the elliptical billiard. The orbits of the elliptical billiard may be classified according to two different main features: those that have an elliptical caustic and those with a hyperbolic caustic (on the phase space of the elliptical billiard, these last orbits belong to invariant curves surrounding the elliptic period–2 point). The invariant curves associated to the orbits with elliptical caustic are easily destroyed by the perturbation (in the same way it happens in the Bunimovich stadium). This is not so easy for those with hyperbolic caustic. So orbits that have a chance to remain elliptic, after perturbation, must be close to those trajectories with hyperbolic caustic.

On the other hand, it is known [3] that the elliptic character of an orbit can be given by a relation between the curvature of the boundary at the impact points and the total length of the trajectory. Since bounces on the straight parts of the elliptical stadium only change the length of the trajectory, the elliptic behavior will depend fundamentally on the number of impacts with the elliptical parts. But, if $a < \sqrt{2}$, no trajectory with hyperbolic caustic, on the elliptical billiard, can have more than two consecutive bounces on the same half-ellipse [2]. Since while bouncing on the half-ellipse, any trajectory on the elliptical stadium billiard behaves exactly as a trajectory on the elliptical table, we must look for periodic orbits, close to orbits of the elliptical billiard with hyperbolic caustic and bouncing twice at the elliptical part of the stadium.

Between all the periodic orbits with this behavior, the pantographic have the following properties:
a) they exist for every even period $p \geq 4$ and so can be studied as the period goes to infinity.
b) they can be explicitly localized and linearized.

Strengthening our choice of this family, in the numerical simulations we have carried out, those pantographic orbits appear as the last ones having *observable* KAM-like-islands.

Proposition 1 *$Pan(i, a, h)$ exists for every $1 < a < \sqrt{2}$ and $h > 0$.*

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Proof: Let P be the point of $Pan(i, a, h)$ located in the right half-ellipse and on the first quadrant. Let $\lambda \in [0, \pi/2]$ be such that $P = (a \cos \lambda + h, \sin \lambda)$, and let $\beta > 0$ be the angle of the trajectory by P, with the normal to the boundary. Using the obvious symmetries, it is easy to see that P is a point of $Pan(i, a, h)$ if the straight line passing by P with slope $\tan(\pi/2 - 2\beta)$ cuts the y-axis at $(0, -i)$. (see figure 3)
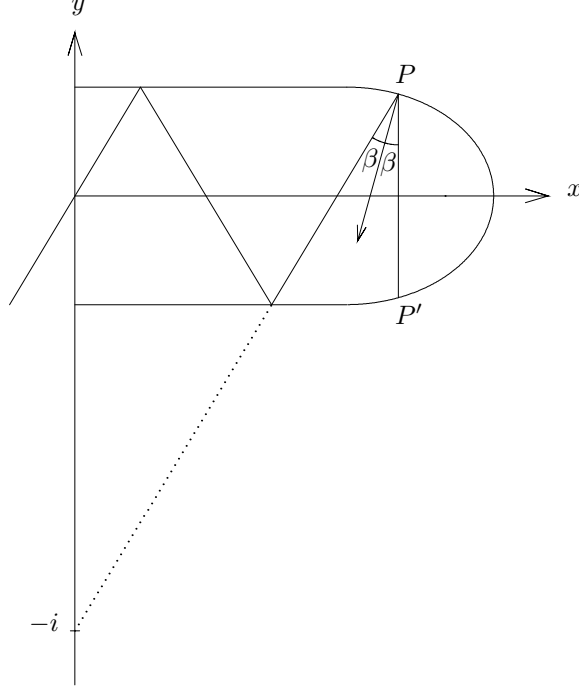

Figure 3:

So,

$$\tan 2\beta = \frac{h + a \cos \lambda}{i + \sin \lambda}.$$

And since $\tan \beta = \cos \lambda / (a \sin \lambda)$ we have

$$a \tan \lambda - \frac{1}{a \tan \lambda} = \frac{2(i\sqrt{1 + \tan^2 \lambda} + \tan \lambda)}{h\sqrt{1 + \tan^2 \lambda} + a} \tag{1}$$

It follows from the same arguments that a trajectory containing the vertical piece from $P' = (a \cos \lambda + h, -\sin \lambda)$ to $P = (a \cos \lambda + h, \sin \lambda)$ will cut the y-axis at $(0, -y)$, where

$$y = y(t) = \frac{a^2 t^2 - 1}{2at\sqrt{1 + t^2}}(h\sqrt{1 + t^2} + a) - \frac{t}{\sqrt{1 + t^2}}, \tag{2}$$

with $t = \tan \lambda$. Then finding a solution of (1) is equivalent to find a t such that $y(t) = i$.

It is not difficult to verify that $\lim_{t \to 0} y(t) = -\infty$, $\lim_{t \to \infty} y(t) = +\infty$ and

$$\frac{dy}{dt} = \frac{(1 + a^2 t^2)(a + h(1 + t^2)\sqrt{1 + t^2})}{2 a t^2 (1 + t^2)^{\frac{3}{2}}} > 0 \text{ for all fixed } a > 0 \text{ and } h > 0.$$

Then, as $y(t)$ is a continous strictly increasing function running from $-\infty$ to $+\infty$ as t runs from 0 to ∞, $y(t) = i$ has a unique solution $t_i(a, h)$ for each integer i, for every given $a > 0$, $h > 0$ and so (1) has also a unique solution $t_i(a, h) = \tan(\lambda_i(a, h))$.

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On the other hand, as we know [2] that for $a < \sqrt{2}$ no trajectory crossing the $x - axis$ between any two consecutive hits with the boundary can have three consecutive impacts on the same half-ellipse, we conclude that the next impact after P' and P on the vertical piece of the trajectory as described above must be on the straight part of the billiard.

Then if $s(\lambda_i)$ is the arclength corresponding to the point $P = (h + a\cos\lambda_i, \sin\lambda_i)$ of the stadium, and $0 < \beta(\lambda_i) = \arctan 1/(at_i) < \pi/2$ the orbit of $(s(\lambda_i), \beta(\lambda_i))$ under the billiard map T is $Pan(i, a, h)$. So is the orbit of $(s(\lambda_i), \pi - \beta(\lambda_i))$. $\qquad\square$

Remark: Using the same ideas, the existence of those $Pan(i, a, h)$ can be proved for $1 < a < 2$, but we are only interested on $a < \sqrt{2}$.

Lemma 2 *Given i, a and h, let $\lambda_i = \lambda_i(a, h)$ be the solution of (1). Then λ_i goes to $\arctan 1/a$ and $\beta(\lambda_i)$ goes to $\pi/4$ as h goes to ∞; λ_i goes to $\pi/2$ as h goes to 0 for all $1 < a < \sqrt{2}$.*

Proof: The right side of equation (1) goes to zero as $h \to \infty$ so we must have in this limit, $a\tan\lambda = 1/(a\tan\lambda)$ and it follows that $\tan\lambda \to 1/a$ for all i.

To study the behavior when $h \to 0$ it is sufficient to study for $i = 0$, because it follows from the proof of Proposition 1 that if $i < j$ then $t_i < t_j$. For $i = 0$, equation (1) can be rewritten as:

$$\frac{h}{a}(a^2 - \cot^2\lambda)\sqrt{1 + \cot^2\lambda} = [(2 - a^2) + \cot^2\lambda]\cot\lambda \qquad (3)$$

and when $a < \sqrt{2}$, $h \to 0$, $[(2 - a^2) + \cot^2\lambda]\cot\lambda \to 0$ and $t_i(0) \to \pi/2$. Thus, $t_i \to \pi/2$ as $h \to 0$. $\qquad\square$

Remark: Let us call *pantographic-like* orbits on the elliptical billiard the periodic trajectories that have vertical segments both at left and right extremes. It would be amazing to compare the results in Proposition 1 with the existence of those pantographic-like orbits. As can be seen in [4], the $2n$-periodic pantographic-like orbit exists if $a > \overline{a}_n$ where $\overline{a}_n$ satisfies $\tan \pi/n = 2\sqrt{\overline{a}_n^2 - 1}/(\overline{a}_n^2 - 2)$. For instance, there is no 4-periodic pantographic-like orbit if $a < \sqrt{2}$, or 6-periodic if $a < 2$.

3 Hyperbolicity of the Pantographic Orbits

Proposition 3 *For each i, let $\alpha_i = \sqrt{\frac{2+2i}{2+i}}$.*

   1. *For $i \geq 0$ if $\alpha_i < a < \sqrt{2}$, there exists a unique $h_i(a)$ such that if $h < h_i(a)$, $Pan(i, a, h)$ is elliptic and if $h > h_i(a)$, $Pan(i, a, h)$ is hyperbolic.*

   2. *For $i \geq 1$, if $1 < a < \alpha_i$, then $Pan(i, a, h)$ is hyperbolic for all $h > 0$.*

Proof: For fixed $a < \sqrt{2}$, i and h, let $\lambda_i(a, h)$ be the solution of (1), β be the angle, with the normal, of the outgoing trajectory at $P = (a\cos\lambda_i + h, \sin\lambda_i)$ and s the corresponding arclength. So $T^{4+2i}(s, \beta) = (s, \beta)$ and to study the stability of this orbit we must analyze the eigenvalues of $\mathcal{D}T^{4+2i}|_{(s,\beta)}$.

Let (s_n, β_n) and (s_{n+1}, β_{n+1}) be two consecutive impacts of a trajectory with the two different half-ellipses (with $k \geq 0$ impacts with the straight parts between them), or two consecutive impacts of a trajectory with the same half-ellipse (with $k = 0$), then (see, for instance,[3])

$$\mathcal{D}T^{k+1}|_{(s_n, \beta_n)} = \frac{(-1)^k}{\cos\beta_{n+1}}\begin{pmatrix} l_{n,n+1}K_n - \cos\beta_n & l_{n,n+1} \\ K_nK_{n+1}(l_{n,n+1} - \cos\beta_n - \cos\beta_{n+1}) & l_{n,n+1}K_{n+1} - \cos\beta_{n+1} \end{pmatrix}$$

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where K stands for the curvature, and $l_{n,n+1}$ is the total length of the trajectory between the two impacts with the half-ellipses.

Then, using elementary geometry and the symmetries of the trajectory, we can write $\mathcal{D}T^{4+2i}|_{(s,\beta)} = (M_1 M_2)^2$ with

$$M_j = \frac{1}{\cos \beta} \begin{pmatrix} l_j K - \cos \beta & l_j \\ K^2 (l_j - 2 \cos \beta) & l_j K - \cos \beta \end{pmatrix}$$

and where $l_1 = 2 \sin \lambda_i$, $l_2 = 2\sqrt{(h + a \cos \lambda_i)^2 + (i + \sin \lambda_i)^2}$ and $K = a/(a^2 \sin^2 \lambda_i + \cos^2 \lambda_i)^{3/2}$.

Now if we define

$$\Delta_i(a, h) = \left(\frac{l_1 K}{\cos \beta} - 1 \right) \left(\frac{l_2 K}{\cos \beta} - 1 \right),$$

we have that $Pan(i, a, h)$ is *elliptic* if $0 < \Delta_i(a, h) < 1$, *parabolic* if $\Delta_i(a, h) = 0$ or 1 and *hyperbolic* if $\Delta_i(a, h) < 0$ or $\Delta_i(a, h) > 1$, which means that the eigenvalues of $\mathcal{D}T^{4+2i}|_{(s,\beta)}$ are respectively purely imaginary and unitary, equal to 1, real and one bigger than 1 and the other smaller than 1 (remember that the system is conservative).

To study the function $\Delta_i(a, h)$, we need the following lemma:

Lemma 4 *The function $\Delta_i(a, h)$ has the following properties for $h > 0$ and $1 < a < \sqrt{2}$:*

1. $\Delta_i(a, h) > 0$

2. $\frac{\partial \Delta_i}{\partial h} > 0$

3. $\lim_{h \to +\infty} \Delta_i(a, h) = +\infty$

4. $\lim_{h \to 0} \Delta_i(a, h) = L_i(a) = (\frac{2}{a^2} - 1)(\frac{2(i+1)}{a^2} - 1) > 0$

Proof of the lemma: If $a < \sqrt{2}$ the half-osculating circles of the ellipse are entirely contained inside the ellipse [4], and so $l_1 > \frac{\cos \beta}{K}$. Since $l_2 > l_1$, Δ_i is the product of two positive factors and property 1 follows.

To prove property 2, we remark first that, from formula 2 we can derive implicitly $\partial t / \partial h$ and show that it is negative.

We have

$$\frac{l_1 K}{\cos \beta} = \frac{2}{a^2 \sin^2 \beta + \cos^2 \beta} = 2 \frac{1 + t^2}{1 + a^2 t^2}$$

and this is a decreasing function of t if $a > 1$. So the first factor of Δ_i is a decreasing function of t.

For $0 < \lambda < \pi/2$ $(t > 0)$ K is a decreasing function of λ, and so a decreasing function of t. As $\tan \beta = 1/at$, $\cos \beta$ is an increasing function of t. This implies that $K/\cos \beta$ decreases with t. Now

$$\frac{\partial l_2^2}{\partial \lambda} = \frac{-a \sin \lambda}{1 + \sin \lambda} (\tan 2\beta - \tan \beta)$$

is negative, as $\beta < \pi/4$.

Thus, Δ_i is a product of two decreasing functions of t, and so, is an increasing function of h which is property 2.

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Property 3 is obvious since $l_2 \to \infty$ as $h \to \infty$ and all the other quantities are bounded.

When $h \to 0$, $\lambda_i \to \pi/2$, $l_1 \to 2$, $\cos\beta \to 1$, $K \to 1/a^2$ and $l_2 \to 2(1+i)$, which implies property 4. □

Now we finish the proof of Proposition 3. Given i, we know from the lemma above that for each $1 < a < \sqrt{2}$, $\Delta_i(a, h)$ is an increasing function of h running from $L_i(a)$ to ∞. The function $L_i(a)$ decreases with a, $L_i(1) = 1 + 2i > 1$ if $i > 0$ and $L_i(\sqrt{2}) = 0$ for every i. $L_i = 1$ has the solution $a = \sqrt{2}\sqrt{\frac{1+i}{2+i}} = \alpha_i$.

So, if $\alpha_i < a < \sqrt{2}$, there exists a unique $h_i(a)$ such that $\Delta_i(a, h_i(a)) = 1$ and $h < h_i(a)$ implies $\Delta_i(a, h_i(a)) < 1$, $h > h_i(a)$ implies $\Delta_i(a, h_i(a)) > 1$.

On the other hand, if $a < \alpha_i$, $\Delta_i(a, h) > 1$ for all h and the result follows. □

For each $i \geq 0$ fixed, and for all $\alpha_i < a < \sqrt{2}$, formula (1) and $\Delta_i(a, h) = 1$ constitute a system equivalent to

$$a^2 \sin^3 \lambda + \frac{i}{2}(a^2 - 1)\sin^2 \lambda - \sin \lambda - \frac{i}{2} = 0 \qquad (4)$$

$$h_i(a) = \frac{a\sqrt{1 - \sin^2 \lambda}}{(a^2 + 1)\sin^2 \lambda - 1}\left(2i\sin\lambda + 1 - (a^2 - 1)\sin^2 \lambda\right) \qquad (5)$$

where $\lambda = \lambda_i(a)$.

The values $\sin \lambda = 1/a$ and $h = \sqrt{a^2 - 1}$ satisfy the equations above for $i = 0$. So, $h_0(a) = \sqrt{a^2 - 1}$ for $1 < a < \sqrt{2}$.

Now, let be $y = \sin \lambda_i(a)$. The problem of finding $h_i(a)$ is reduced to finding a root of the cubic polynomial

$$P_{i,a}(y) = y^3 + \frac{i}{2a^2}(a^2 - 1)y^2 - \frac{1}{a^2}y - \frac{i}{2a^2}$$

in the interval $(0, 1)$. $P_{i,a}(y) = 0$ can be rewritten as

$$y\left(y^2 - \frac{1}{a^2}\right) = \frac{i}{2a^2}\left(1 - (a^2 - 1)y^2\right)$$

The left hand side is a cubic polinomial with roots at $0, \pm 1/a$; it is positive for $y > 1/a$ and negative in $(0, 1/a)$. The right hand side is a quadratic polynomial, with roots $\pm 1/\sqrt{a^2 - 1}$ and which is negative for $y > 1/\sqrt{a^2 - 1} > 1/a$. This implies that $P_{i,a}$ has only one positive real root and that this root belongs to $(1/a, 1/\sqrt{a^2 - 1})$. As the left hand side is 0 for $y = 1/a$ and the right hand side is positive, $P_{i,a}(1/a) < 0$. However, $P_{i,a}(1) > 0$ for $a > \alpha_i$ and $P_{i,a}(1) < 0$ for $a > \alpha_i$. This implies that for each i, $P_{i,a}(y)$ has one and only one real root in $(0, 1)$ for $\alpha_i < a < \sqrt{2}$.

This root can be found by standard techniques:

$$y_i(a) = 2\sqrt{A}\cos\theta - \frac{i(a^2 - 1)}{6a^2} \qquad (6)$$

where

$$A = \frac{12a^2 + (a^2 - 1)^2 i^2}{(6a^2)^2}$$

$$B = \frac{2\left(18a^2(1 + 2a^2)i + (a^2 - 1)^3 i^3\right)}{(6a^2)^3}$$

$$\cos 3\theta = -B/(2A^{3/2})$$

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Moreover, a more careful investigation shows that $B/(2A^{3/2}) < 1$. When $B/(2A^{3/2}) < -1$, there is only one real root and one should make use of the definition of cosine for imaginary arguments so "cos" is changed into "csh". For $-1 < B/(2A^{3/2}) \leq 1$, there are 3 real roots and we choose $0 \leq \theta \leq \pi/3$, in order to have $y_i(a) > 0$.

One can also write (6) as

$$y_i(a) = \frac{A}{C^{\frac{1}{3}}} + C^{\frac{1}{3}} - \frac{(-1 + a^2)\, i}{6\, a^2}$$

where $C = (\sqrt{B^2 - 4A^3} - B)/2$, and when it is imaginary, the choice of the logarithm branch is the same as the choice of θ above.

If we introduce this value of $\sin \lambda_i(a)$ in (5), we obtain explicit formulae for the $h_i(a)$. These functions are plotted in figure 4.
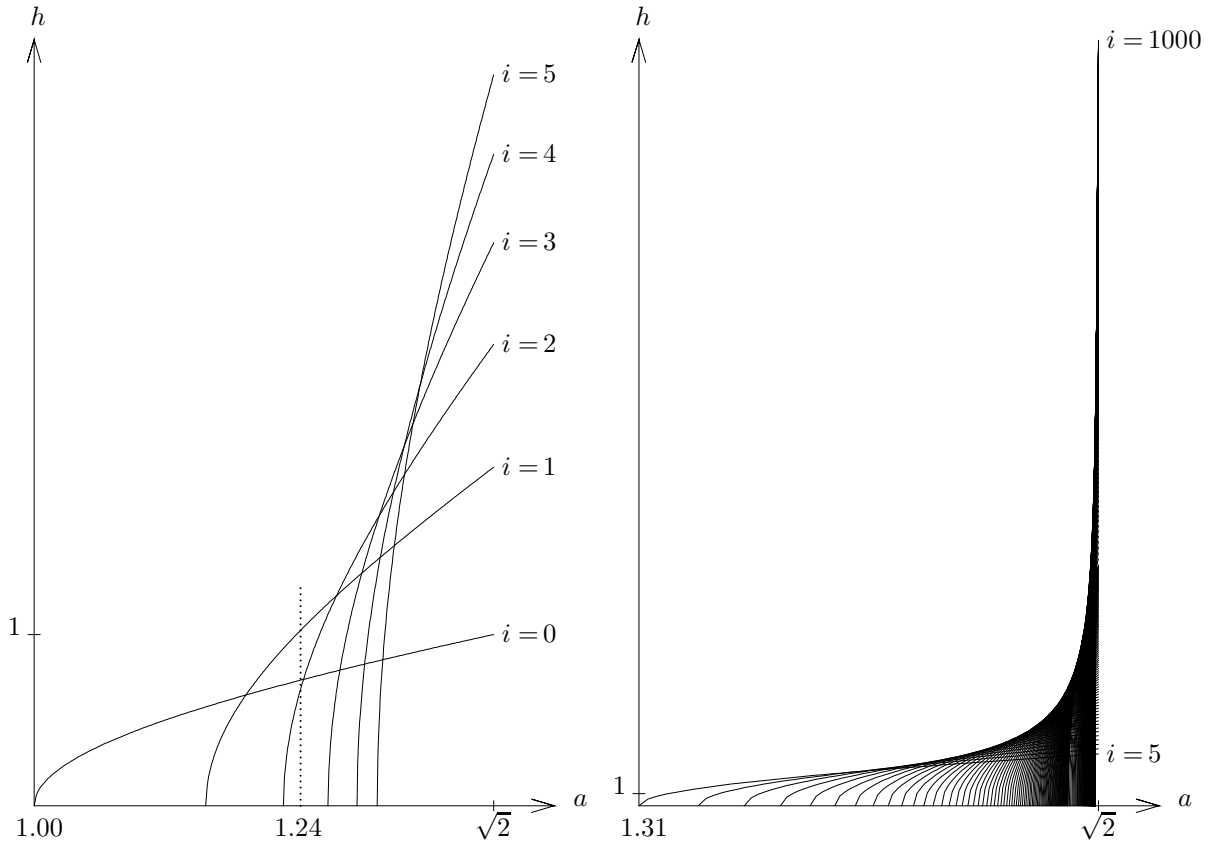

Figure 4: Graphs of h_i

4   A lower bound for the chaotic zone

For $i = 0$, $h_0(a) = \sqrt{a^2 - 1}$ is a strictly increasing function in $(1, \sqrt{2})$. Figure 4 shows that this is also true for $i = 1,\ 2$. For $i \geq 3$ we have the following:

Proposition 5 *For each fixed $i \geq 3$, $\frac{dh_i}{da} > 0$.*

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Proof: For $i > 0$, we have

$$h_i(a) = \frac{2\,a\,y\,\sqrt{1-y^2}}{-1+(1+a^2)\,y^2}\left(i+\frac{-1+a^2\,y^2}{i}\right)$$

where $y = y_i(a)$ is the only root of $P_{i,a}(y)$ in $(1/a, 1)$. Then, we have that

$$\frac{dy}{da} = \frac{-a\,y^2\,(i+2\,y)}{i\,y\,(a^2-1)+3\,a^2\,y^2-1} < 0.$$

Now

$$\frac{dh_i}{da} = \frac{\partial h_i}{\partial a} + \frac{\partial h_i}{\partial y}\frac{dy}{da}$$

can be written as

$$\frac{dh_i}{da} = \frac{\partial h_i}{\partial a}\left(1+\frac{a}{y(1-y^2)}\frac{dy}{da}\right) + \left(\frac{\partial h_i}{\partial y} - \frac{a}{y(1-y^2)}\frac{\partial h_i}{\partial a}\right)\frac{dy}{da}.$$

As

$$\frac{\partial h_i}{\partial a} = \frac{2\,y\,\sqrt{1-y^2}\,\left(1-y^2-2\,a^2\,y^2+3\,a^2\,y^4+a^4\,y^4-i^2\,(1-y^2+a^2\,y^2)\right)}{i\,(-1+y^2+a^2\,y^2)^2}$$

$$\frac{\partial h_i}{\partial y} = \frac{2\,a\,\left(1-y^2-2\,a^2\,y^2+3\,a^2\,y^4+a^4\,y^4+2\,a^2\,y^4-2\,a^2\,y^6-2\,a^4\,y^6-i^2(1-y^2+a^2\,y^2)\right)}{i\,\sqrt{1-y^2}\,(-1+y^2+a^2\,y^2)^2}$$

we obtain that

$$\frac{\partial h_i}{\partial y} - \frac{a}{y(1-y^2)}\frac{\partial h_i}{\partial a} = \frac{-4\,a^3\,y^4}{i\,\sqrt{1-y^2}\,(-1+y^2+a^2\,y^2)} < 0$$

and

$$1+\frac{a}{y(1-y^2)}\frac{dy}{da} = \frac{1+y^2\,(3\,a^2\,y^2-1-a^2)+i\,y\,(1-y^2+a^2\,y^2)}{-(1-y^2)\,(i\,y\,(a^2-1)+3\,a^2\,y^2-1)} < 0$$

Now we will show that $\partial h_i/\partial a < 0$ for $i \geq 3$, so $dh_i/da > 0$ for $i \geq 3$.

$$\frac{\partial h_i}{\partial a} = \frac{2\,y\,\sqrt{1-y^2}}{i\,(-1+y^2+a^2\,y^2)^2}\left(D-i^2\,E\right)$$

where

$$\begin{aligned}
D &= (1-y^2)+a^2\,y^2\,(3y^2-1)+a^2\,y^2\,(a^2\,y^2-1) > 0 \\
E &= 1+(a^2-1)\,y^2 > 0
\end{aligned}$$

Now

$$E \geq 1+\frac{y^2}{3} \geq 1+\frac{1}{6} = \frac{7}{6} > 0$$

as $\sqrt{2} \geq a \geq \alpha_1 = 2/\sqrt{3}$, for $i \geq 1$, and $1 \geq y \geq 1/a \geq 1/\sqrt{2}$.

On the other hand

$$0 < D \leq \frac{1}{2}+2a^2\,y^2+a^2\,y^2 = \frac{1}{2}+3a^2\,y^2 \leq \frac{1}{2}+6 = \frac{13}{2}\,.$$

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So if

$$i \geq 3 > \sqrt{\frac{39}{7}} \geq \sqrt{\frac{D}{E}}$$

$\partial h_i / \partial a < 0$ and the result follows. $\qquad\square$

We were also able to found the asymptotical behavior of the h_i at $a = \sqrt{2}$.

Proposition 6 *If $a = \sqrt{2}$ then $\lim\limits_{i \to \infty} \dfrac{h_i^2(\sqrt{2})}{4i} = 1$.*

Proof: For $a = \sqrt{2}$ we have

$$P_{i,\sqrt{2}}(y) = y^3 + \frac{i}{4}y^2 - \frac{1}{2}y - \frac{i}{4}$$

so its only positive real root goes to 1 as i goes to ∞. It follows that

$$\lim_{i \to \infty} \sin \lambda_i(\sqrt{2}) = 1 \ .$$

Since $P_{i,\sqrt{2}}(\sin \lambda) = 0$,

$$\sin \lambda \left(\sin^2 \lambda - \frac{1}{2} \right) = \frac{i}{4} \cos^2 \lambda \ .$$

The left hand side goes to $1/2$ as i goes to ∞ implying

$$\lim_{i \to \infty} \frac{i}{2} \cos^2 \lambda_i(\sqrt{2}) = 1 \ .$$

As

$$h_i^2(\sqrt{2}) = \frac{2 \cos^2 \lambda}{3 \sin^2 \lambda - 1} \left(2i \sin \lambda + \cos^2 \lambda \right)^2$$

using the limits above, gives the desired result. $\qquad\square$

Proposition 6 tells us that there exists an N such that if $i > N$, $h_i^2(\sqrt{2})/4 \approx i$. Plotting $h_i^2(\sqrt{2})/4 \times i$, we have observed the same linear behavior also for small values of i.

For each $1 < a < \sqrt{2}$ there exists a j such that $\alpha_{j-1} \leq a < \alpha_j$. So, $Pan(i, a, h)$ is hyperbolic for $i \geq j$ and is elliptic if $0 \leq i < j$ and $h < h_i(a)$. There exists, then, only a finite number of h_i's defined for this value of a and we can define the announced lower bound by $H(a) = \max_{i<j} \{h_i(a)\}$. Proposition 6 implies that $H(a) \to \infty$ as $a \to \sqrt{2}$. $H(a)$ can be seen on figure 4.

5 Final remarks

Clearly above the curve $H(a)$ all pantographic orbits are hyperbolic and bellow it some have eigenvalues in the unit complex circle. Generically, KAM theory establishes the existence of positive measure elliptic islands surrounding those orbits and our numerical simulations corroborate this result. In this paper we have not *proved* the existence of such islands.

On the other hand, above this bound, although we can prove that all the orbits of this family are hyperbolic, we can not assure the non existence of other elliptic periodic orbits that could be surrounded by positive measure sets of invariant curves and having vanishing Lyapunov exponents. As a matter of

fact, we have observed other periodic orbits of $T_{a,h}$ but the elliptic islands around them seem to disappear for values of $h < H(a)$.

A numerical case study which seems generic for this problem is presented in figures 5, 6 and 7. There, we show the phase space associated for $a = 1.24$ fixed and different values of h. For this value of a we have that $\alpha_2 = \sqrt{1.5} < 1.24 < \sqrt{1.6} = \alpha_3$ (see fig. 4) so $Pan(i, 1.24, h)$ are hyperbolic for any $h > 0$ and $i \geq 3$. The pantographic orbits of period 4, 6 and 8 ($Pan(0, 1.24, h)$, $Pan(1, 1.24, h)$ and $Pan(2, 1.24, h)$) are the only relevant pantographic orbits as they are elliptic for small h. In fact we have $h_0 \approx 0.7332$, $h_1 \approx 1.0236$, $h_2 \approx 0.6770$, $h_2 < h_0 < h_1 = H(1.24)$.

Note that for very small h (in this example, 0.1), a very rich structure of elliptic islands (the white holes on fig. 5) can be observed; some of these islands correspond to other periodic orbits.
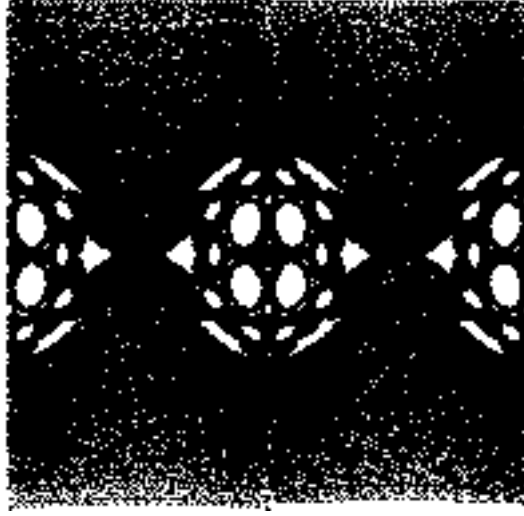

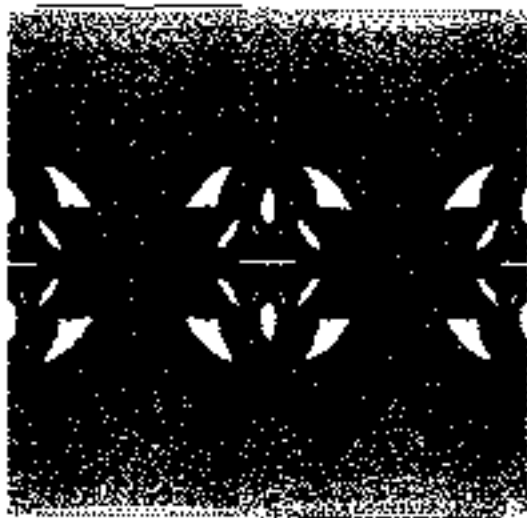Figure 5: $a = 1.24$, $h = 0.1$. 150,000 iterations of a single initial condition.

As h is increased (fig. 6), they gradually disappear and only the pantographic islands seem to remain at $h = 0.45$. Then, at $h = 0.73 > h_2$, the eight islands around $Pan(2, 1.24, h)$ have disappeared and at $h_0 < h = 0.75 < h_1$ only the six islands around $Pan(1, 1.24, h)$ can be seen, since $Pan(0, 1.24, h)$ has also become hyperbolic. In each case, the region outside the elliptic islands seems to be a single ergodic component as it is filled up by a single orbit.

For $h = 1.05 > h_1 = H(1.24)$ (fig. 7) the system seems to be ergodic.

As we can see on figure 4, the order of extinction of the pantographic elliptic islands depends on the value of a. As shown above, for $a = 1.24$, the order is $i = 2$, then $i = 0$ and then $i = 1$. But, for values of a a little bigger than 1.24, it can be 0,2,1 or even 0,1 and 2.

However one can learn from figure 4, that as a approaches $\sqrt{2}$ the last islands to disappear correspond to orbits with long period. This is expressed in proposition 6. So in this system, and this fact seems also to occur in other problems, the obstruction to ergodicity seems to be existence of elliptic islands around orbits with arbitrarily large period and so the existence of an arbitrarily large number of very small islands. Those islands, even though summing up into a positive measure region, can be invisible in a numerical simulation.
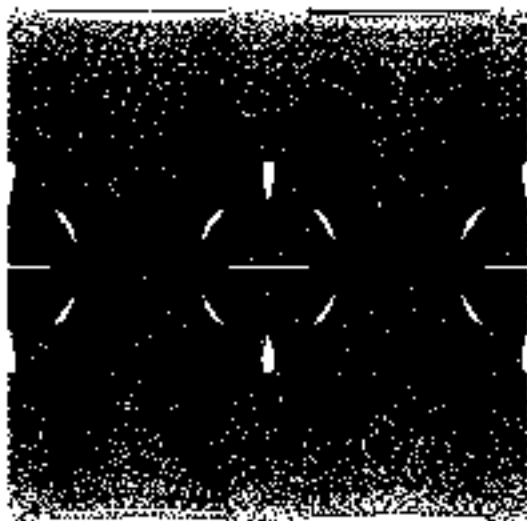
We would like to point out that the pantographic orbits also seem to be the most important ones from the point of view of their focusing properties. In [4], the existence of a positive Lyapunov exponent

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$h = 0.45$
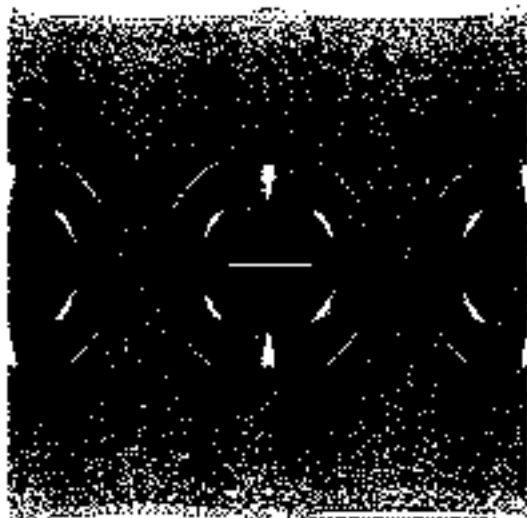
$h = 0.73$

$h = 0.75$

Figure 6: $a = 1.24$ and different h's. On the left side, 150,000 iterations of a single initial condition and. On the right, the iteration of a few initial conditions close to the elliptic pantographic orbits.
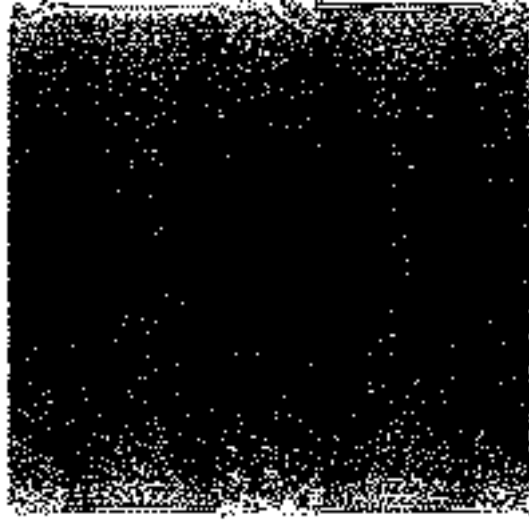
Figure 7: $a = 1.24$, $h = 1.05$. 150,000 iterations of a single initial condition.

for the elliptical billiard was obtained through the study of the behavior of the caustic pencil, or the tangent vector to the invariant curves of the elliptical billiard. In the pantographic orbits, after hitting a half-ellipse twice, in the vertical portion of the trajectory, it crosses to the other half-ellipse. At this moment the focusing distance of the caustic pencil can be very large. This lack of focalization should be compensated by a larger traveling distance, so a bigger h, in order to have a splitting of neighboring trajectories. However, as a approaches $\sqrt{2}$ this focusing distance may tend to ∞. The loss of ellipticity by all the pantographic orbits caused by increasing h indicates that the behavior of the caustic pencil may be controlled at this point and one should be able to prove the existence of positive Lyapunov exponents.

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